SOLUTION TO A REVERSE PROBLEM IN THERMOELASTICITY BY ELECTRICAL SIMULATION

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Devices are discussed which make it feasible to extend the application of electrical simulators and use them for solving reverse problems in thermoelasticity.

In accelerated cyclic heat-resistance testing of structures one must determine the heating mode of some designed element from the given magnitude of thermal stresses. It then becomes necessary to solve a reverse problem in thermoelasticity, where the time characteristic of thermal stresses at some point in the structural element is given and the boundary conditions of the first, of the second, or of the third kind are to be determined.

Just as the solution of reverse problems in heat conduction [1], the solution here also involves certain difficulties. When solving a reverse problem in thermoelasticity, one finds it necessary to impose some limitations on the given stress–time characteristic. These aspects will be discussed here, with regard to solving one such reverse problem in thermoelasticity by electrical simulation.

Let the structural element, its shape and dimensions shown in Fig. 1, consist of thermally thin cover plates 1 and a supporting girder 2 made of different materials and joined together. This structural element is heated symmetrically at both cover plates. It will be assumed that, for example, the heating is effected under boundary conditions of the third kind, with the ambient temperature T and the heat-transfer coefficient α being known functions of time.

Between the plates and the I-beam there takes place contactive heat-transfer characterized by the coefficient α_C.

If T_1 and T_2 are the solutions to the equation of heat conduction

$$\frac{\partial T_1}{\partial \tau} = a_1 \frac{\partial^2 T_1}{\partial x^2} + \frac{\alpha}{c_1 \gamma_1 \delta_1}(T - T_1) - \varkappa \frac{\alpha_C}{c_1 \gamma_1 \delta_1}(T_1 - T_2), \tag{1}$$

$$\frac{\partial T_2}{\partial \tau} = a_2 \frac{\partial^2 T_2}{\partial x_i^2} + \nu \frac{\alpha_C}{c_2 \gamma_2 \delta_2}(T_1 - T_2), \quad (i = 1,\ 2), \tag{2}$$

where

$$\varkappa = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant d, \\ 0 & \text{for} \quad d < x \leqslant l, \end{cases} \qquad \nu = \begin{cases} 1 & \text{for} \quad i = 1, \\ 0 & \text{for} \quad i = 2 \end{cases}$$

with the initial and the boundary conditions

$$T_1(x,\ 0) = T_2(x_i,\ 0) = T_0,$$

$$\frac{\partial T_1(0,\ \tau)}{\partial x} = \frac{\partial T_1(l,\ \tau)}{\partial x} = \frac{\partial T_2(d,\ \tau)}{\partial x_1} = \frac{\partial T_2(h,\ \tau)}{\partial x_2} = 0,$$

$$\frac{\partial T_2(0,\ \tau)}{\partial x_1}\delta_2 - \frac{\partial T_2(0,\ \tau)}{\partial x_2}\delta_3 = T_2|_{x_1=0} - T_2|_{x_2=0} = 0,$$

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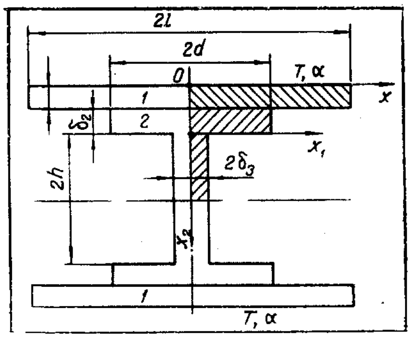

Fig. 1. Designed element.

then in the two-dimensional problem of thermoelasticity the conditions of equilibrium and compatibility with respect to strains in a plate 1 and in the girder 2

$$\delta_1 \int_0^l \sigma_1 dx + \delta_2 \int_0^d \sigma_2 dx_1 + \delta_3 \int_0^h \sigma_2 dx_2 = 0, \ \varepsilon_1 = \varepsilon_2 = \varepsilon$$

yield the following expressions for the thermal stresses in both components 1 and 2

$$\sigma_i = E_i \Delta - \alpha_i E_i \theta_i, \tag{3}$$

where

$$\Delta = \frac{\delta_1 \alpha_1 E_1 \int_0^l \theta_1 dx + \delta_2 \alpha_2 E_2 \int_0^d \theta_2 dx_1 + \delta_3 \alpha_2 E_2 \int_0^h \theta_2 dx_2}{\delta_1 l E_1 + \delta_2 d E_2 + \delta_3 h E_2},$$

$$\theta_i = T_i - T_0.$$

The time characteristics of T, α, and the thermal stresses σ_1, which have been calculated according to Eq. (3) for a point of the structural element at section $x = l$, are shown in Fig. 2a.

Let it be required to accelerate the test, i.e., to shorten the time heat-resistance test cycles under the same extreme levels of thermal stresses. An example of such an accelerated stress cycling has been selected here, on some rational basis, for a point at $x = l$ and is depicted by curve 3 in Fig. 2b.

In order to determine the needed time characteristic of the ambient temperature $T = T(\tau)$ for given values of the heat-transfer coefficient $\alpha = \alpha(\tau)$, we used an electrical model shown schematically in Fig. 3.

The electrical model represents a closed-loop static system of automatic regulation which includes an RC network for simulating the geometrical dimensions and the thermophysical properties of the material of the designed element, a heat-transfer simulator (HTS) for selecting the appropriate kind of boundary conditions, and a stress simulator (ST) for establishing the necessary transient thermal stresses at the point under study. The regulation system includes also an amplifier (A), to the input of which are applied uniscale bipolar signals of the reference-levels and the actual-level stress in the designed element.

The actual stress level is determined according to formula (3) by the method of finite difference approximations. Here resistors $R_1, \ldots, R_n$ synthesize the first term on the right-hand side of the equation, the inverter (INV) and resistor R_m synthesize the second term on the right-hand side, the summator (SU) adds its input signals, and the measuring device (MD) records the boundary conditions.

In order to establish the stability limits of the regulation system, we have analyzed a simpler version of it. The RC network was represented here by three aperiodic first-order terms.

In this case, with the amplifier and the inverter treated as linear devices, the transfer function of the open-loop regulation system can be written as follows:

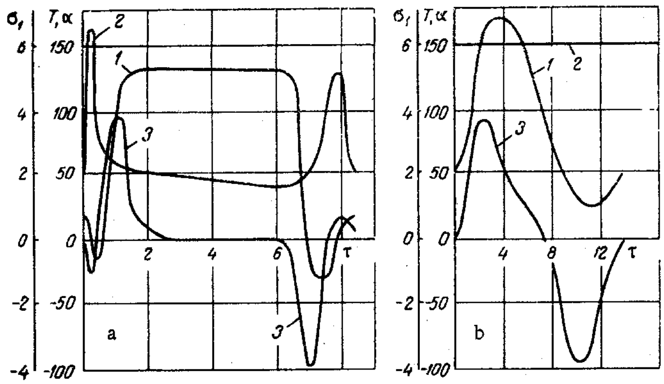

Fig. 2. Original cycle (a) and accelerated cycle (b): 1) T; 2) α; 3) σ. Time $\tau \cdot 10^{-3}$ sec in (a) and $\tau \cdot 10^{-2}$ sec in (b).

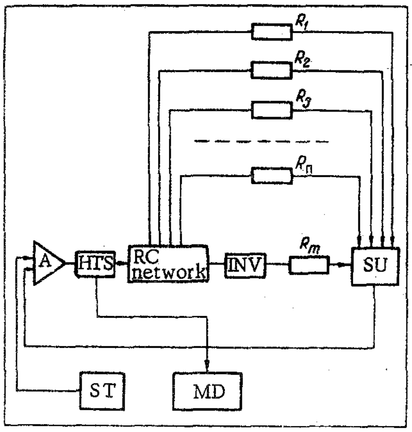

Fig. 3. Structural diagram of the electrical model.

$$W(p) = k_0 [W_1 k_1 + W_1 W_2 k_2 + W_1 W_2 W_3 (k_3 - k_4)].$$

Considering that the transfer function for an aperiodic component is

$$W_i = \frac{1}{1 + m_i p},$$

we obtain

$$W(p) = \frac{k_0 [k_1 (1 + m_2 p)(1 + m_3 p) + k_2 (1 + m_3 p) + k_3 - k_4]}{N},$$

where

$$N = (1 + m_1 p)(1 + m_2 p)(1 + m_3 p).$$

For the transfer function of the closed-loop system we have

$$W^*(p) = \frac{W(p)}{1 + W(p)}.$$

The characteristic equation will be written as

$$k_0 [k_1 (1 + m_2 p)(1 + m_3 p) + k_2 (1 + m_3 p) + k_3 - k_4] + N = 0.$$

Inserting here the numerical values of the parameters, we obtain

$$4.3p^3 + k_0 (12.4p^2 + 4.9p) + 18.4p^2 + 17.6p + 1 = 0.$$

Using Routh's algebraic criterion, we find that the system remains dynamically stable with an amplifier gain within the 10-1000 range. An additional check by Mikhailov's criterion confirms that the system with such parameter values is stable.

We will now discuss the limitation which must be imposed on the input signal simulating the thermal stress σ.

The transform of the output signal in a closed-loop regulation system is

$$u(p)_{\text{in}} = W(\overset{*}{p})\, u(p)_{\text{out}} \tag{4}$$

Let a harmonic signal

$$u(p)_{\text{in}} = M_\sigma \sigma(p) \tag{5}$$

from the stress simulator (SU) appear at the input. From relations (4) and (5) we have

$$\Delta u(p) = u(p)_{\text{out}} - u(p)_{\text{in}} = M_\sigma \sigma(p) \frac{1}{1 + W(p)}. \tag{6}$$

The transform $\Delta u(p)$ happens to be the solution to the reverse problem and, therefore, the voltage at the amplifier output is related to the ambient temperature $T(\tau)$. The magnitude of $T(\tau)$ is always limited for some reason, i.e., $T = T_{max} f_1(\omega\tau)$ with $|f_1(\omega\tau)| \leq 1$.

In view of this, the maximum voltage at the amplifier input can be expressed as

$$\Delta u_{\max} = \frac{T_{\max}}{M_T k_0}.$$

When $\sigma = \sigma_{max} f_2(\omega\tau)$ with $|f_2(\omega\tau)| \leq 1$, then expression (6) yields

$$\Delta u \geqslant M_\sigma \sigma_{\max} V$$

or

$$\sigma_{\max} \leqslant \frac{T_{\max}}{M_T M_\sigma k_0 V}, \tag{7}$$

where $V = |1/(1 + W(i\omega))|$.

Inequality (7) defines the limit which must be imposed on σ_{max}, in order to ensure the existence of a solution to this reverse problem in thermoelasticity. This limit is not only related to T_{max} but also to the frequency $\omega = 2\pi/\tau_{cy}$, where τ_{cy} denotes the duration of one heating cycle, and to the boundary conditions for V in terms of $m_1 = (R' + R_\alpha)C'$. Here $R_\alpha = M_\alpha/\alpha(\tau)$ and, when the boundary conditions are of the third kind, one must let $\alpha(\tau) = \alpha_{max}$ for an estimate of σ_{max}.

The proposed electrical model is to some extent universal with respect to boundary conditions and, therefore, will yield a solution also for boundary conditions of the first or the second kind. In the first case $m_1 = (R' + R_q)C'$, where R_q denotes some arbitrary constant resistance, and the thermal flux $q(\tau)$ is proportional to the output current from the amplifier (A).

An example of a solution to this particular reverse problem in thermoelasticity obtained by the proposed scheme of electrical simulation is shown in Fig. 2b.

Curve 1 represents the ambient temperature $T(\tau)$ corresponding to a given stress variation $\sigma(\tau)$ (curve 3) at a constant heat-transfer coefficient α (curve 2).

The accuracy of the solution to this reverse problem was evaluated on a digital computer by the method of elementary balances in the forward problem, with the ambient temperature $T(\tau)$ found on the electrical model. The relative error of calculated stresses did not exceed 6% of those stipulated in the reverse problem.

NOTATION

T	is the ambient temperature, °C;
α	is the heat-transfer coefficient, $W/m^2 \cdot deg$;
α_c	is the coefficient of contactive heat transfer;
τ	is the time, sec;
T_i, a_i, c_i, γ_i $(i = 1, 2)$	are the temperature, the thermal diffusivity, the specific heat, and the specific weight of a plate $(i = 1)$ and of the girder $(i = 2)$, respectively;
x, x_i $(i = 1, 2)$	are space coordinates;
l, d, h, δ_i $(i = 1, 2, 3)$	are the geometrical dimensions of the structural element;
$\sigma_i, E_i, \alpha_i, \varepsilon_i$ $(i = 1, 2)$	are the thermal stress (dyn/mm^2), the elasticity modulus, the thermal expansivity, and the unit strain of a plate $(i = 1)$ and of the girder $(i = 2)$, respectively;
W_i, k_i, m_i $(i = 1, 2, 3)$	are the transfer function, the switching coefficient, and the time constant of an aperiodic component;
k_0	is the amplifier gain;
p	is the Laplace operator;
$W(p)$	is the transfer function of an open-loop system;
$W^*(p)$	is the transfer function of a closed-loop system;
V	is the modulus of frequency characteristic;
$u(p)_{in}$	is the transform of an input signal;
$u(p)_{out}$	is the transform of an output signal;
$\sigma(p)$	is the transform of thermal stress;
M_σ, M_T, M_α	are the scale factors for the stress, the temperature, and the heat-transfer coefficient, respectively;
R', C'	are the resistance and the capacitance which simulate the geometrical dimensions and the thermophysical properties of a plate.

LITERATURE CITED

1. V. E. Prokof'ev, Inzh.-Fiz. Zh., 22, No. 2, 310 (1972).